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VICTOR KLEE

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13. ABSTRACT

Due in considerable measure to its connection with linear programming, the combinatorial study of convex polytopes has progressed greatly in the past twenty-five years and many outstanding problems have been solved. However, the d-step conjecture, first formulated by W. M. Hirsch in 1957, remains unsettled. Both for its intrinsic interest and for its relationship to questions of computational complexity, it is probably the most important open problem on the combinatorial structure of polytopes. This report presents several equivalent forms of the d-step conjecture. Some deal explicitly with edge-paths on polytopes, one involves matrix pivot operations, and one concerns an exchange process for simplicial bases.

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HOW MANY STEPS?

Victor Klee

Stimulated since the early 1950's by its relationship to linear programming and more recently by other connections with computational questions, the combinatorial study of convex polytopes has advanced greatly in the last twenty-five years. For an overview the reader may skim successively the survey articles or books of Klee 1966 [14], Grünbaum 1967 [8], Grünbaum and Shephard 1969 [11], Grünbaum 1970 [9], McMullen and Shephard 1971 [21], Grünbaum 1975 [10], Klee 1975 [15] and Barnette 1978 [3] —— and then read the recent proofs by Billera and Lee [4] and Stanley [23] of an important 1971 conjecture of McMullen [20].

The d-step conjecture was first formulated by W. M. Hirsch in 1957 (see Dantzig [5,6]), and despite progress on many other fronts it remains unsettled. Because of intrinsic interest and connections with questions of computational complexity, and because solutions may well require the development of deep new methods, the d-step conjecture and its relatives are probably the most important open problems on the combinatorial structure of convex polytopes. The present note, which is extracted from a longer article in preparation, states several equivalent forms of the d-step conjecture. Some deal explicitly with edge-paths on polytopes, one involves matrix pivot operations, and one concerns an exchange process for simplicial bases.

Suppose that X is the nonnegative orthant R_+^d in real d-space R^d, x is the point $(0,0,\ldots,0)$, x' is the point $(1,1,\ldots,1)$, and $X' = x' - R_+^d$, a translate of the nonpositive orthant $-R^d_+$. Then

(a) X and X' are affine orthants with respective vertices $x \in \text{int } X'$ and $x' \in \text{int } X$, and

(b) the intersection $P = X \cap X'$ is bounded.

One form of the d-step conjecture asserts that <u>whenever conditions</u> (a) and (b) <u>are satisfied the vertices</u> x <u>and</u> x' <u>of the polytope</u> P <u>can be joined by a path formed from at most</u> d <u>edges of</u> P. That is certainly true in the example, for there P is the unit cube $[0, 1]^d$. However, even when (a) and (b) are augmented by the requirement that P is simple (each vertex incident to precisely d edges), there are many other possibilities for P. In particular, the possible number of vertices then ranges from $d^2 - d + 2$ to $4\binom{3k-1}{k}$ when $d = 2k$ and to $2\binom{3k+1}{k}$ when $d = 2k + 1$.

The d-polytopes of the above form $X \cap X'$ have precisely $2d$ facets (faces of dimension $d-1$). Hence a formal strengthening of the above conjecture is the conjecture that $\Delta(d, 2d) = d$, where $\Delta(d, n)$ denotes the maximum diameter of d-polytopes P with n facets. (That is, $\Delta(d, n)$ is the smallest integer k such that any two vertices x and x' of such a P can be joined by a path formed from at most k edges of P.) A further formal strengthening is the conjecture that $\Delta(d,n) \leq n-d$. However, it was proved by Klee and Walkup [17] that these conjectures are all equivalent, though not necessarily on a dimension-for-dimension bases. Another equivalent conjecture is that <u>any two vertices of a simple polytope can be joined by a path that does not revisit any facet</u> [17].

Turning now to matrix pivot operations, we note that the d-step conjecture is equivalent to the following:

If the real $d \times (2d+1)$ *matrices* $A = (I, B, c)$ *and* $A' = (B', I, c')$ *are row-equivalent, where* $d \geq 2$, I *is the* $d \times d$ *identity matrix, and the columns* c *and* c' *are* > 0, *and if the polyhedron*

$$P = \{x \in R_+^{2d} : (I,B)x = c\}$$

is bounded, then it is possible to pass from A *to* A' *by a sequence of* $\leq d$ *feasible pivots followed if necessary by a permutation of rows.*

Here a *pivot*, as applied to an $m \times (n+1)$ matrix $S = (s_{ij})$, is the operation of choosing (i,j) with $j \leq n$ and $s_{ij} \neq 0$, then dividing the i^{th} row of S by s_{ij} so as to obtain 1 in position (i,j), and finally subtracting appropriate multiples of the i^{th} row from other rows so as to obtain 0 in all positions (h,j) for $h \neq i$. A pivot is *feasible* if the last column of the matrix is nonnegative both before and after the pivot.

Of the several forms of the d-step conjecture presented here, this is closest to Hirsch's original form [5, pp. 160 and 168] and most closely related to linear programming methods. In the example below, $d = 2$ and the pairs (i,j) under the arrows indicate the positions of the pivot entries.

$$\begin{array}{ccccc} 1 & 0 & 2 & -1 & 2 \\ 0 & 1 & -1 & 2 & 2 \\ \\ I & & B & & C \end{array} \xrightarrow[(1,3)]{} \begin{array}{ccccc} 1/2 & 0 & 1 & -1/2 & 1 \\ 1/2 & 1 & 0 & 3/2 & 3 \\ \\ & & & & \end{array} \xrightarrow[(2,4)]{} \begin{array}{ccccc} 2/3 & 1/3 & 1 & 0 & 2 \\ 1/3 & 2/3 & 0 & 1 & 2 \\ \\ & B' & & I & C' \end{array}$$

A set $B \subset R^{d-1}$ is a *simplicial basis* (also called a *minimum positive* basis) for R^{d-1} if it is the vertex-set of a $(d-1)$ - simplex whose interior includes the origin. Equivalently, B is affinely independent, is of cardinality d, and the origin 0 is a strictly positive combination of the points of B. Another equivalent form of the d-step conjecture is reminiscent of the exchange process used to show all linear bases of a vector space are of the same cardinality. It is as follows:

If B *and* B' *are disjoint simplicial bases of* R^{d-1} *and the union* $U = B \cup B'$ *is a Haar set (every set of* $d-1$ *points* of U is linearly independent), *then there is a sequence* $B = B_0, B_1, \ldots, B_d = B'$ *of simplicial bases such that for* $1 \leq i \leq d$, B_i *is obtained from* B_{i-1} *by replacing a point of* B_{i-1} *with a* point of $U \sim B_{i-1}$.

In the example below, $d = 3$ and $0 < \varepsilon < 1$. The rows represent points of B_i.

$$\underset{B_o = B}{\begin{matrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{matrix}} \longrightarrow \underset{B_1}{\begin{matrix} 1 & 0 \\ 0 & 1 \\ -\varepsilon & -1 \end{matrix}} \longrightarrow \underset{B_2}{\begin{matrix} 1 & 1-\varepsilon \\ 0 & 1 \\ -\varepsilon & -1 \end{matrix}} \longrightarrow \underset{B_3 = B^1}{\begin{matrix} 1 & 1-\varepsilon \\ -1 & -\varepsilon \\ -\varepsilon & -1 \end{matrix}}$$

As evidence in favor of the d-step conjecture, one might consider the facts that it is obvious when $d \in \{2,3\}$, is easily proved when $d = 4$ [13], and has been proved for $d = 5$ [15]. In fact, the result for $d = 5$ has been extended by Adler and Dantzig [1] to a much more general class of combinatorial structures. Also, the conjecture has been proved, for arbitrary d, for polytopes arising from certain sorts of linear programs (see [15] for references, and see especially Provan and Billera [22]).

As evidence against the d-step conjecture, we note that when stated without the boundedness condition (b), it is correct when $d \in \{2,3\}$ but not when

$d = 4$ [17]. Other strengthened forms of the conjecture have been disproved by Walkup [25], Mani and Walkup [19], and Todd [24].

It seems likely that the conjectue is false for $d = 12$ and perhaps even for $d = 6$. if that is so, what can be said about the asymptotic behavior of $\Delta(d,2d)$ as $d \to \infty$? Does $\Delta(d,2d)$ increase linearly with d? (The known counterexamples [19, 24, 25] to strengthened forms of the conjecture seem to be only "linearly bad".) Quadratically? Polynomially? Exponentially? Any of these conclusions would be of great interest. If it could be shown that $\Delta(d,2d)$ is bounded above by a polynomial in d, the resulting insight might lead to a new pivot rule for the simplex method of linear programming which would combine the practical advantages of Dantzig's pivot rule with the theoretical advantages of the Shor-Khachian algorithm. (Dantzig's algorithm is excellent in the practical sense [5,7], but its worst-case behavior is exponentially bad [16]. The Shor-Khachian algorithm is "good" in the sense of being polynomially bounded [12], but is not a useful practical tool in its present form [7].) If it could be shown that $\Delta(d,2d)$ increases exponentially with d, that would indicate a strong limitation on the worst-case efficiency of any edge-following algorithm for linear programming.

Though sharper results are known for a few small values of d and $n-d$, (see [15] for references) the best general bounds on $\Delta(d,n)$ are the following, due respectively to Adler [1] and Larman [18]:

$$\left\lfloor (n-d) - \frac{(n-d)}{\lfloor 5d/4 \rfloor} \right\rfloor + 1 \le (d,n) \le 2^{d-3} n$$

In particular, $d \le \Delta(d,2d) \le 2^{d-3} d$.

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